

Inversion Complexity of Functions of Multi-Valued Logic

V. V. Kochergin*, A. V. Mikhailovich†

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Abstract

The minimum number of NOT gates in a logic circuit computing a Boolean function is called the inversion complexity of the function. In 1957, A. A. Markov determined the inversion complexity of every Boolean function and proved that $\lceil \log_2(d(f) + 1) \rceil$ NOT gates are necessary and sufficient to compute any Boolean function f (where $d(f)$ is the maximum number of value changes from greater to smaller over all increasing chains of tuples of variables values). This result is extended to k -valued functions computing in this paper. Thereupon one can use monotone functions “for free” like in the Boolean case. It is shown that the minimum sufficient number of non-monotone gates for the realization of the arbitrary k -valued logic function f is equal to $\lceil \log_2(d(f) + 1) \rceil$ if Post negation (function $x + 1 \pmod k$) is used in NOT nodes and is also equal to $\lceil \log_k(d(f) + 1) \rceil$, if Łukasiewicz negation (function $k - 1 - x$) is used in NOT nodes. Similar extension for another classical result of A. A. Markov for the inversion complexity of a system of Boolean functions to k -valued logic functions has been obtained.

Keywords: multi-valued logic functions, logic circuits, circuit complexity, nonmonotone complexity, inversion complexity, Markov’s theorem.

Let P_k be the set of all functions of k -valued logic and M be the set of all functions that are monotone relative to order $0 < 1 < \dots < k -$

*Lomonosov Moscow State University (Faculty of Mechanics and Mathematics, Bogoliubov Institute for Theoretical Problems of Microphysics); vvkoch@yandex.ru

†National Research University Higher School of Economics; anna@mikhaylovich.com

1. We will investigate the complexity of the realization of k -valued logic functions by circuits [1] (also known as combinational machine or circuits of computation [2]) over bases B of the form:

$$B = M \cup \{\omega_1, \dots, \omega_p\}, \quad \omega_i \in P_k \setminus M, \quad i = 1, \dots, p,$$

where the weight of any function from M equals zero, the weight of function ω_i , $i = 1, \dots, p$, equals 1.

Let us denote the sum of the weights of the elements from circuit S by *non-monotone complexity* $I_B(S)$ of circuit S over basis B . In other words it is the number of circuit elements corresponding to non-monotone basis functions. Let $f \in P_k$, $F \subseteq P_k$. We denote the minimum non-monotone complexity of the circuit that realizes function f (system F respectively) by *non-monotone complexity* $I_B(f)$ of function f (*complexity* $I_B(F)$ of the system F respectively) over basis B .

We emphasize two natural bases — basis B_P that consists of all non-monotone functions and Post negation ($x + 1 \pmod k$), and basis B_L that consists of all non-monotone functions and Łukasiewicz negation ($k - 1 - x$). We will use the term *inversion complexity* that is similar to the Boolean function case [3, 4] because of these two bases, although it is slightly unsuitable.

A.A. Markov [3, 4] obtained the exact inversion complexity value for an arbitrary Boolean function or a Boolean function system over basis $B_0 = M \cup \{\bar{x}\}$ [3, 4] (the exact statement of this result is given below). E.I. Nechiporuk [6] obtained the exact inversion complexity value for an arbitrary Boolean function realization by a Boolean formula (this result was reobtained much later in [7, 8]). Some results dealt with the inversion complexity can be also found in [9–13]. In this paper classical Markov's results are extended to the case of k -valued logic functions. The presentation of the results corresponds with the presentation of Markov's results in [14].

The set $\{0, 1, \dots, k - 1\}$ is denoted by E_k . A sequence of tuples

$$\tilde{\alpha}_1 = (\alpha_{11}, \dots, \alpha_{1n}), \quad \tilde{\alpha}_2 = (\alpha_{21}, \dots, \alpha_{2n}), \quad \dots, \quad \tilde{\alpha}_r = (\alpha_{r1}, \dots, \alpha_{rn})$$

from the set E_k^n is called *an increasing chain with respect to order* $0 < 1 < \dots < k - 1$ or just *chain*, if all tuples $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r$ are different and the following inequalities hold

$$\alpha_{ij} \leq \alpha_{i+1,j}, \quad i = 1, \dots, r - 1, \quad j = 1, \dots, n.$$

The tuples $\tilde{\alpha}_1$ and $\tilde{\alpha}_r$ are called *initial* and *terminal* tuples of the chain respectively.

Let $f(x_1, \dots, x_n)$ be a function of k -valued logic. An ordered pair of tuples $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \dots, \beta_n)$, $\tilde{\alpha}, \tilde{\beta} \in E_k^n$, is called a *jump* for the function f , if

- 1) $\alpha_j \leq \beta_j$, $j = 1, \dots, n$;
- 2) $f(\tilde{\alpha}) > f(\tilde{\beta})$.

A *jump for a system of functions* is a pair of tuples which is a jump for any function of the system.

Let $F = \{f_1, \dots, f_m\}$, $m \geq 1$, be a system of k -valued logic function with arguments x_1, \dots, x_n . Let C be a chain of the form

$$\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r.$$

Decrease $d_C(F)$ of the system F over chain C is the number of jumps for the system F of the form $(\tilde{\alpha}_i, \tilde{\alpha}_{i+1})$.

Decrease $d(F)$ of the system F is the maximum $d_C(F)$ over all chains C .

Now we can give the exact statement for the Markov's classical result [3, 4]. Let F be a system of Boolean functions. Then $I_{B_0}(F) = \lceil \log_2(d(F) + 1) \rceil$.

Let

$$d(B) = \max\{d(\omega_1), \dots, d(\omega_p)\}.$$

Theorem 1. *Let F be a system of k -valued logic functions. Then*

$$I_B(F) \geq \lceil \log_{d(B)+1}(d(F) + 1) \rceil.$$

First we prove an auxiliary statement.

Lemma 1. *Let F be a system of k -valued logic functions. Then*

$$d(F) \leq (d(B) + 1)^{I_B(F)} - 1.$$

Proof. Let $F = \{f_1, \dots, f_m\}$, $m \geq 1$, be a set of functions of k -valued logics with arguments x_1, \dots, x_n . The proof is by induction on $I_B(F)$.

If $I_B(F) = 0$ the all functions from F are monotone. Hence, $d(F) = 0$.

Assume that the assertion is valid for any $G \subset P_k$ such that $I_B(G) \leq I_B(F) - 1$. Consider circuit S with n inputs x_1, \dots, x_n which realizes function system F and contains exactly $I_B(F)$ elements of unit weight. Let us select the first such vertex (according to any correct numeration) and denote the corresponding gate by E . Gate E corresponds to t -place function ω , $\omega \in \{\omega_1, \dots, \omega_p\}$. Denote by $h_1(x_1, \dots, x_n), \dots, h_t(x_1, \dots, x_t)$ functions that are given at the inputs of E . Denote by S' a circuit that is obtained from the

circuit S by replacement of gate E with one more input with variable y . The circuit S' realizes system $G = \{g_1, \dots, g_m\}$ with the following properties:

$$f_i(x_1, \dots, x_n) = g_i(\omega(h_1(x_1, \dots, x_n), \dots, h_t(x_1, \dots, x_n)), x_1, \dots, x_n), \\ i = 1, \dots, m.$$

Moreover, $I_B(G) \leq I_B(F) - 1$.

Consider a chain

$$C = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r)$$

such that $d(F) = d_C(F)$.

Let us consider the sequence C' of $(n+1)$ -tuples:

$$(\omega(h_1(\tilde{\alpha}_1), \dots, h_t(\tilde{\alpha}_1)), \tilde{\alpha}_1), \dots, (\omega(h_1(\tilde{\alpha}_r), \dots, h_t(\tilde{\alpha}_r)), \tilde{\alpha}_r).$$

The sequence C' is not a chain, but it can be split into p parts (each part consists of consecutive elements from C') C'_1, \dots, C'_p such that each C'_j , $j = 1, \dots, p$, is a chain and p satisfies the inequalities $1 \leq p \leq d(B) + 1$.

By the induction assumption relation

$$d_{C'_i}(G) \leq d(G) \leq (d(B) + 1)^{I_B(G)} - 1 = (d(B) + 1)^{I_B(F) - 1} - 1$$

is valid for all j , $j = 1, \dots, p$. Now, using equalities

$$f_i(\tilde{\alpha}) = g_i(\omega(h_1(\tilde{\alpha}), \dots, h_t(\tilde{\alpha})), \tilde{\alpha}), \quad i = 1, \dots, m,$$

we get

$$d_C(F) \leq \sum_{i=1}^p d_{C'_i}(G) + p - 1 \leq \sum_{i=1}^p ((d(B) + 1)^{I_B(F) - 1} - 1) + p - 1 \leq (d(B) + 1)^{I_B(F)} - 1.$$

Thus, Lemma 1 is proved. \square

Proof of the Theorem 1. Lemma 1 implies the inequality

$$d(F) \leq (d(B) + 1)^{I_B(F)} - 1.$$

$I_B(F)$ is an integer. Thus, we obtain the necessary estimation. Thus, Theorem 1 is proved.

Remark. The estimation from Theorem 1 is approximate even if $k = 2$. Indeed, let us consider system $F = \{\bar{x}, \bar{y}\}$. The decrease of the system equals 2. While any circuit, that uses only one non-monotone element, realizes a two-argument function with decrease of 1. Thus, the inversion complexity of the system cannot equal 1 in any basis.

Now we pass on to the upper bound estimation. Let $f_1(x_1, x_2, \dots, x_n), \dots, f_s(x_1, x_2, \dots, x_n)$ be a tuple of k -valued logic functions. A function $g(z_1, \dots, z_s, x_1, x_2, \dots, x_n)$, such that

$$\begin{aligned} g(1, 0, \dots, 0, x_1, x_2, \dots, x_n) &= f_1(x_1, x_2, \dots, x_n), \\ g(0, 1, \dots, 0, x_1, x_2, \dots, x_n) &= f_2(x_1, x_2, \dots, x_n), \\ &\vdots \\ g(0, \dots, 0, 1, x_1, x_2, \dots, x_n) &= f_s(x_1, x_2, \dots, x_n) \end{aligned}$$

is called *s*-connector for the tuple $f_1(x_1, x_2, \dots, x_n), \dots, f_s(x_1, x_2, \dots, x_n)$.

A set of *s*-connectors for a set of *s*-tuples of functions (one *s*-connector for each *s*-tuple) is called *s*-connector for the set.

Lemma 2. *Let B be a basis of the form $B = M \cup \{\omega(x_1, \dots, x_q), \omega \in P_k \setminus M, q \geq 1\}$. Let $F_1 = \{f_{11}, \dots, f_{s1}\}, \dots, F_M = \{f_{1m}, \dots, f_{sm}\}$ be arbitrary set of *s*-tuples of k -valued logic functions. Then there is an *s*-connector G of the set such that*

$$I_B(G) \leq \max\{I_B(F_1), \dots, I_B(F_s)\}.$$

Proof. The proof is by induction on $r = \max\{I_B(F_1), \dots, I_B(F_s)\}$.

If $r = 0$ then the functions from $F_i, i = 1, \dots, s$, are monotone. Then let G be the following set:

$$\{g_j \mid g_j = \max(\min(\phi(z_1), f_{1j}), \dots, \min(\phi(z_s), f_{sj}), j = 1, \dots, m\},$$

where

$$\phi(z) = \begin{cases} k-1, & \text{if } z \neq 0; \\ 0, & \text{elsewhere.} \end{cases}$$

Let $r > 0$ (induction step). Denote by $S_i(\tilde{x})$ any circuit with inputs x_1, x_2, \dots, x_n that realizes the function system $F_i, i = 1, \dots, s$, which contains $\max\{I_B(F_i), 1\}$ gates, corresponding to function ω . Let us select the first vertex (according to any correct numeration) corresponding to the function ω in circuit $S_i(\tilde{x})$. Denote by $h_{i1}(x_1, \dots, x_n), \dots, h_{iq}(x_1, \dots, x_n)$ functions that are given at the inputs of the gate. Denote by S' a circuit with inputs y, x_1, x_2, \dots, x_n which is obtained from the circuit S by replacing the selected gate with one more input with variable y . Denote by $f'_{ij}(y, x_1, x_2, \dots, x_n), j = 1, \dots, m$, functions that are realized at the outputs of circuit $S_i(y, \tilde{x})$. Then

$$\begin{aligned} f_{ij}(x_1, x_2, \dots, x_n) &= \\ f'_{ij}(\omega(h_{i1}(x_1, x_2, \dots, x_n), \dots, h_{iq}(x_1, x_2, \dots, x_n)), x_1, x_2, \dots, x_n), \\ &\quad j = 1, \dots, m. \end{aligned}$$

Suppose $F'_i = \{f'_{i1}, \dots, f'_{im}\}$. Since $I_B(F'_i) \leq r - 1$, $i = 1, \dots, s$, by the induction assumption there is a set of functions

$$G' = \{g'_j(z_1, \dots, z_s, y, x_1, x_2, \dots, x_n) \mid j = 1, \dots, m\},$$

such that

$$\begin{aligned} I_B(G') &\leq \max\{I_B(F'_1), \dots, I_B(F'_s)\} \leq r - 1; \\ g'_j(1, 0, \dots, 0, y, x_1, x_2, \dots, x_n) &= f'_{1j}(y, x_1, x_2, \dots, x_n), \quad j = 1, \dots, m; \\ g'_j(0, 1, \dots, 0, y, x_1, x_2, \dots, x_n) &= f'_{2j}(y, x_1, x_2, \dots, x_n), \quad j = 1, \dots, m; \\ &\vdots \\ g'_j(0, 0, \dots, 1, y, x_1, x_2, \dots, x_n) &= f'_{sj}(y, x_1, x_2, \dots, x_n), \quad j = 1, \dots, m. \end{aligned}$$

Let us replace variable y by function

$$\begin{aligned} Y(z_1, \dots, z_s, x_1, x_2, \dots, x_n) &= \\ \omega(\max(\min(\phi(z_1), h_{11}(x_1, x_2, \dots, x_n)), \dots, \min(\phi(z_s), h_{s1}(x_1, x_2, \dots, x_n))), \dots, \\ &\quad \max(\min(\phi(z_1), h_{1q}(x_1, x_2, \dots, x_n)), \dots, \min(\phi(z_s), h_{sq}(x_1, x_2, \dots, x_n)))) \end{aligned}$$

in function $g'_j(z_1, \dots, z_s, y, x_1, x_2, \dots, x_n)$, $j = 1, \dots, m$,

Since equalities

$$\begin{aligned} Y(1, 0, \dots, 0, x_1, x_2, \dots, x_n) &= \omega(h_{11}(x_1, x_2, \dots, x_n), \dots, h_{1q}(x_1, x_2, \dots, x_n)), \\ Y(0, 1, \dots, 0, x_1, x_2, \dots, x_n) &= \omega(h_{21}(x_1, x_2, \dots, x_n), \dots, h_{2q}(x_1, x_2, \dots, x_n)), \\ &\vdots \\ Y(0, 0, \dots, 1, x_1, x_2, \dots, x_n) &= \omega(h_{s1}(x_1, x_2, \dots, x_n), \dots, h_{sq}(x_1, x_2, \dots, x_n)) \end{aligned}$$

are valid, we get that function

$$\begin{aligned} g_j(z_1, \dots, z_s, x_1, x_2, \dots, x_n) &= \\ g'_j(z_1, \dots, z_s, Y(z_1, \dots, z_s, x_1, x_2, \dots, x_n), x_1, x_2, \dots, x_n) \end{aligned}$$

is s -connector for the tuple f_{1j}, \dots, f_{sj} , $j = 1, \dots, m$. Moreover, there are inequalities $I_B(G) \leq 1 + I_B(G') \leq r$ for the set $G = \{g_1, \dots, g_m\}$.

Lemma 2 is proved. \square

Let $f(x_1, \dots, x_n)$ be an arbitrary k -valued logic function, $C = (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_r)$ be an arbitrary chain of tuples from E_k^n . Denote by $u_C(f)$ the maximum length of subsequence $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_t$ of sequence C such that $f(\tilde{\beta}_1) > f(\tilde{\beta}_2) > \dots > f(\tilde{\beta}_t)$.

Inversion power $u(f)$ of the function f is the maximum $u_C(f)$ over all chains C from E_k^n . Obviously, for any function f the inequalities $1 \leq u(f) \leq d(f) + 1$ hold. Moreover, if function f is not monotone then $u(f) \geq 2$.

Inversion power $u(B)$ of basis B is the maximum $u(f)$ over all functions f from B .

Theorem 2. *Let F be a system of k -valued logic functions. Then*

$$I_B(F) \leq \lceil \log_{u(B)}(d(F) + 1) \rceil.$$

Proof. Let $u(B) = s$. Suppose $\omega(x_1, \dots, x_q) \in B$ such that $u(\omega) = s$. Let $B' = M \cup \{\omega(x_1, \dots, x_q)\}$. Since $I_{B'}(F) \geq I_B(F)$ it is enough to prove the inequality $I_{B'}(F) \leq \lceil \log_s(d(F) + 1) \rceil$. The proof is by induction on $R(F) = \lceil \log_s(d(F) + 1) \rceil$.

If $R(F) = 0$, then $d(F) = 0$. Hence, all the functions from F are monotone. Thus, $I_B(F) = 0$.

For the induction step let G be a set of functions such that $R(G) \leq R(F) - 1$. Suppose the Theorem statement is correct for G .

Denote by T_1 a set of n -tuples of elements from E_k such that for any chain C with terminal tuple from T_1 the inequality $d_C(F) < s^{R(F)-1}$ holds, that is

$$T_1 = \{\tilde{\alpha} \in E_k^n \mid d_C(F) < s^{R(F)-1} \text{ for any chain } C \text{ with terminal tuple } \tilde{\alpha}\}.$$

Further, denote by T_i , $i = 2, \dots, s-1$, a set of n -tuples with elements from E_k such that for any chain of elements from $E_k^n \setminus (T_1 \cup \dots \cup T_{i-1})$ with a terminal tuple from T_i inequality $d_C(F) < s^{R(F)-1}$ holds, that is

$$T_i = \{\tilde{\alpha} \in E_k^n \setminus (T_1 \cup \dots \cup T_{i-1}) \mid d_C(F) < s^{R(F)-1} \text{ for any chain } C, \\ C \subset E_k^n \setminus (T_1 \cup \dots \cup T_{i-1}), \text{ with terminal tuple } \tilde{\alpha}\}.$$

Finally, let

$$T_s = E_k^n \setminus (T_1 \cup \dots \cup T_{s-1}).$$

Note that if $\tilde{\alpha} \in T_i$ and $\tilde{\beta} \prec \tilde{\alpha}$ then $\tilde{\beta} \in T_1 \cup \dots \cup T_{i-1}$, $i = 1, \dots, s$.

Now we prove that for any chain C of elements from T_s , the inequality $d_C(F) < s^{R(F)-1}$ also holds. Assume the converse. Hence, there is a chain C_s with initial tuple $\tilde{\alpha}_s$, $\tilde{\alpha}_s \in T_s$, such that $d_{C_s}(F) \geq s^{R(F)-1}$. Since $\tilde{\alpha}_s \notin T_s$, there is a chain C_{s-1} with initial tuple $\tilde{\alpha}_{s-1}$, $\tilde{\alpha}_{s-1} \in T_{s-1}$ and terminal tuple $\tilde{\alpha}_s$, $\tilde{\alpha}_s \in T_s$, such that $d_{C_{s-1}}(F) \geq s^{R(F)-1}$. Similarly, for $i = s-2, \dots, 1$, there is a chain C_i with initial tuple $\tilde{\alpha}_i$, $\tilde{\alpha}_i \in T_i$, and terminal tuple $\tilde{\alpha}_{i+1}$, $\tilde{\alpha}_{i+1} \in T_{i+1}$, such that $d_{C_i}(F) \geq s^{R(F)-1}$.

Then for chain $C = C_1 \cup \dots \cup C_s$ the relations

$$d_C(F) = d_{C_1}(F) + \dots + d_{C_s}(F) \geq s (s^{R(F)-1}) = s^{R(F)} > d(F),$$

hold. This contradicts the definition of $d(F)$.

Let $f_j \in F = \{f_1, \dots, f_m\}$. Suppose

$$f_{ij}(x_1, x_2, \dots, x_n) = \begin{cases} 0, & \text{if } (x_1, x_2, \dots, x_n) \in T_1 \cup \dots \cup T_{i-1}; \\ f_j(x_1, x_2, \dots, x_n), & \text{if } (x_1, x_2, \dots, x_n) \in T_i; \\ k-1, & \text{if } (x_1, x_2, \dots, x_n) \in T_{i+1} \cup \dots \cup T_s; \end{cases}$$

$i = 1, \dots, s$.

Let

$$F_i = \{f_{ij} \mid f_j \in F\}, \quad i = 1, \dots, s.$$

By the definition of the set F_i the inequalities $d(F_i) < s^{R(F)-1}$, $i = 1, \dots, s$, hold. Hence, inequalities

$$d(F_i) \leq s^{R(F)-1} - 1, \quad i = 1, \dots, s,$$

are valid. Thus,

$$R(F_i) = \lceil \log_s(d(F_i) + 1) \rceil \leq \lceil \log s^{R(F)-1} \rceil = R(F) - 1, \quad i = 1, \dots, s.$$

By the definition of the value $s = u(\omega)$ there is a chain $(\beta_{11}, \dots, \beta_{1q}), (\beta_{21}, \dots, \beta_{2q}), \dots, (\beta_{s1}, \dots, \beta_{sq})$, such that $\omega(\beta_{11}, \dots, \beta_{1q}) > \omega(\beta_{21}, \dots, \beta_{2q}) > \dots > \omega(\beta_{s1}, \dots, \beta_{sq})$.

We define functions ξ_1, \dots, ξ_q by the following equalities

$$\xi_j(x_1, \dots, x_n) = \beta_{ij}, \quad i = 1, \dots, s, \quad j = 1, \dots, q,$$

which are valid for all tuples (x_1, \dots, x_n) from T_i .

Let $b_i = \omega(\beta_{11}, \dots, \beta_{1q})$, $i = 1, \dots, s$.

We define functions $\lambda_j(x)$, $j = 1, \dots, k-1$. Let

$$\lambda_j(x) = \begin{cases} 0, & \text{if } x < j; \\ 1, & \text{if } x \geq j. \end{cases}$$

We define functions $\mu_i(x_1, \dots, x_n)$, $i = 1, \dots, s$. Let

$$\mu_i(x_1, \dots, x_n) = \begin{cases} 0, & \text{if } (x_1, \dots, x_n) \in T_1 \cup \dots \cup T_{i-1}; \\ 1, & \text{if } (x_1, \dots, x_n) \in T_i \cup \dots \cup T_s. \end{cases}$$

Note that all these functions are monotone.

Consider s -connector $G = \{g_j(z_1, \dots, z_s, \tilde{x}) \mid j = 1, \dots, m\}$ for the tuple of function $\{(f_{1j}(\tilde{x}), \dots, f_{sj}(\tilde{x})) \mid j = 1, \dots, m\}$. By Lemma 2 there exists such an s -connector.

Replace variable z_i , $i = 1, \dots, s$, by function

$$Z_i(x_1, \dots, x_n) = \min \{ \lambda_{b_i}(\omega(\xi_1(x_1, \dots, x_n), \dots, \xi_q(x_1, \dots, x_n))), \mu_i(x_1, \dots, x_n) \}.$$

in function $g_j(z_1, \dots, z_s, \tilde{x})$, $j = 1, \dots, m$.

Since function $Z_i(x_1, \dots, x_n)$ equals 1 on tuples from T_i and equals 0 on the other tuples, we get that for all tuples (x_1, \dots, x_n) from T_i inequalities

$$g_j(Z_1(x_1, \dots, x_n), \dots, Z_s(x_1, \dots, x_n), x_1, \dots, x_n) = f_{ij}(x_1, \dots, x_n) = f_j(x_1, \dots, x_n), \quad i = 1, \dots, s, \quad j = 1, \dots, m,$$

are valid.

To realize functions Z_1, \dots, Z_s one have used monotone functions gates and one gate corresponding to function ω . By induction assumption we get

$$\begin{aligned} I_{B'}(F) &\leq I_{B'}(G) + 1 \leq \max\{I_{B'}(F_1), \dots, I_{B'}(F_s)\} + 1 \leq \\ &\leq \max\{\lceil \log_s(d(F_1) + 1) \rceil, \dots, \lceil \log_s(d(F_s) + 1) \rceil\} \leq \lceil \log_s s^{R(F)-1} \rceil + 1 = R(F). \end{aligned}$$

That completes induction step.

Theorem 2 is proved. \square

If basis B is such that $d(B) + 1 = u(B)$, Theorem 1 and Theorem 2 give the exact value for non-monotone complexity in basis B for any system of k -valued logic functions. Obviously, this equality holds for bases B_P and B_L .

Theorem 3. *Let F be a system of k -valued logic functions. Then*

$$I_{B_P}(F) = \lceil \log_2(d(F) + 1) \rceil, \quad I_{B_L}(F) = \lceil \log_{k-1}(d(F) + 1) \rceil.$$

A Shannon function for inversion complexity over basis B of n -argument function and a system of m functions are defined in standard way:

$$I_B(n) = \max_{f \in P_k(n)} I_B(f), \quad I_B(n, m) = \max_{F = \{f_1, \dots, f_m\}, f_j \in P_k(n)} I_B(F).$$

Let

$$T(k, n) = (k-1)n - \left\lfloor \frac{(k-1)n}{k} \right\rfloor + 1 = (k-2)n + \left\lceil \frac{n}{k} \right\rceil + 1.$$

Theorem 4. For any n and m , $n \geq 1$, $m \geq 2$, inequalities

$$\begin{aligned} I_{BP}(n) &= \lceil \log_2 T(k, n) \rceil, & I_{BP}(n, m) &= \lceil \log_2((k-1)n+1) \rceil; \\ I_{BL}(n) &= \lceil \log_{k-1} T(k, n) \rceil, & I_{BL}(n, m) &= \lceil \log_{k-1}((k-1)n+1) \rceil. \end{aligned}$$

hold.

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